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Chaos control of chaotic plankton dynamics in the presence of additional food, seasonality, and time delay

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ABSTRACT

In marine ecology, plankton systems have rich and complex dynamics. The control of chaos in plankton systems is one of the main motives of recent research. In this article, the chaos control mechanisms is explored using internal parameters and external forces. Also, the availability of additional food for both zooplankton and fish is incorporated. The present work enhances the existing literature on the chaos control dynamics of available additional food for predator population by focusing on the consequences of predation delay and seasonal perturbations. It presents a comprehensive view of the different ranges of alternative food, periodic fluctuations, and time delay on the dynamics of the proposed system. The stability analysis of the systems with and without predation delay is studied. The Hopf-bifurcation analysis is performed by taking time delay as a bifurcation parameter. Control of chaos in plankton-fish dynamics is explored with respect to additional food for plankton and fish population, predation delay, seasonal perturbations and delayed feedback control. The numerical simulation presents the validation of theoretical results.

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1. Introduction

In marine ecosystems, the plankton dynamics are highly chaotic and unpredictable, which may result in the extinction of some species, the occurrence of plankton blooms, disease outbreaks, or anything unexpected in nature [1]. The natural processes in biological systems are likewise complex, non-linear, and their outcomes can explore chaotic behavior [2–5]. The control of chaos is a challenging matter and mechanisms behind these particular patterns are not fully understood [6]. Originating from the empirical work [7,8] to recent research [2,4,9] on ecological systems, all the research work was about the chaotic behavior of natural communities. The main focus of these studies is plankton communities due to their irreplaceable contribution to the biosphere. Many ecologists have reported the various chaos control techniques in marine kingdom using different biological factors [10–14]. These studies provide a stabilizing influence for which the chaotic dynamics converted to stable limit cycle oscillation. Moreover, various feedback and non-feedback mechanisms are also available in the lit-

erature [15–18], which presents a different view on chaos control strategies in dynamical systems. Gakkhar et al. [19] have studied the impact of non-linear closure terms in plankton dynamics with different functional responses for suppressing chaos. The marine ecosystem is highly affected by seasonal forces, periodic fluctuations, and noise, etc. Many studies [20–24] have explored the role of these external forces on marine kingdom.

The delay-induced mathematical models [25–27] represent more realistic but complex dynamics since a time delay can cause instability in the system by inducing various oscillations and periodic solutions. The impact of different types of time delays (maturation delay, predation delay, toxin liberation delay, and gestation delay, etc.) on various plankton dynamics has been studied in the last two decades [28–33]. But the recent work [34] shows that the time lag can stabilize the plankton system by excluding chaotic dynamics.

The parameters responsible for controlling the unpredictable behavior of the dynamical systems for the co-existence of plankton-fish species have been extensively studied by researchers. Additional food source in aquatic life plays a significant role in the co-existence of plankton-fish dynamical systems and is the main focus of recent research. The natural supply of alternative

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food to predator population can provide biological control [25,26], pest control [27], and disease control [28,29] in the food chain models. In marine ecosystem, alternative food is available for zooplankton and fish species: as in the southwestern Gulf of Maine, mesozooplankton grazers (Semibalanus, C. finmarchicus, etc.) consistently avoided toxic phytoplankton (Alexandrium spp. and other dinoflagellates) and selectively fed on alternative prey [35]. In the northeast Pacific Ocean of the British Columbia coast, zooplankton species not only consume phytoplankton but some hunter zooplankton like krill, jellyfish, and crabs, etc., prefer alternative food (eggs, small zooplankton, and mass of other organisms, etc.) [25,36]. Most of the fish species consume zooplankton, but the fish species (piscivorous fish, apex fish predators, marine mammals, and reptiles, etc.) found in coral reefs located in the Pacific Ocean, the Indian Ocean, the Caribbean Sea, the Red Sea, and the Persian Gulf, prefer planktivorous fish, squid, shellfish, or sea-grass, and a few eat other mammals as additional food [37,38].

Sahoo et al. [9] have analyzed that the available additional food can stabilize the chaotic behavior of the plankton dynamics. They have studied the consequences of providing alternative food to predator zooplankton on plankton dynamics. Extending the work [9], we have formulated a mathematical model in which alternative food is available to the zooplankton and fish population. Sahoo et al. [9] studied the role of quality and quantity of alternative food for controlling chaotic plankton dynamics. But in the present paper, we have studied the impact of total available additional food to zooplankton and fish, seasonal forces, and time delay in suppressing chaotic oscillations of a plankton–fish dynamical system to make ecological balance. We have formulated plankton–fish dynamics consisting of the biomass of phytoplankton (P(t)), zooplankton (Z(t)), and fish (F(t)). We have assumed that additional food is available for both predator populations, viz. zooplankton and fish. The present manuscript is summarized as follows; The non-delayed mathematical model, its dynamical properties, and stability analysis with numerical simulation are discussed in Section 2, Section 3, and Section 4, respectively. The analysis of the chaotic system and its control using available additional food and seasonality is studied in Section 5. The delayed plankton model, its positivity, boundedness, and Hopf-bifurcation analysis with numerical validation are presented in Section 6. The role of time delay in stabilizing chaotic plankton dynamics is discussed in Section 7. Finally, we conclude in Section 8.

2. The mathematical model

Firstly, we consider the following model [9]

$$\left. \begin{aligned} \frac{dX}{dt} &= R_0 X \left(1 - \frac{X}{K_0}\right) - C_1 A_1 \frac{XY}{B_1 + X}, \\ \frac{dY}{dt} &= A_1 \frac{X}{B_1 + X} Y - A_2 \frac{YZ}{B_2 + \alpha \mu A + Y} - D_1 Y, \\ \frac{dZ}{dt} &= C_2 A_2 \frac{(Y + \mu A)Z}{B_2 + \alpha \mu A + Y} - D_2 Z, \end{aligned} \right\} \quad (1)$$

where X, Y, and Z are the densities of prey, middle predator, and top predator. R_0 is the intrinsic growth rate, K_0 is the carrying capacity, A_1 and $C_2 A_2$ are the maximum growth rate of a middle and top predator, respectively. The parameters D_1 and D_2 denote the natural death rate of middle and top predators, respectively. The available additional food for the top predator is denoted by $\alpha \mu A$ and other parameters have their own meaning.

To modify model (1), we have formulated a plankton–fish model (2) in which additional food is available for both predator populations viz., zooplankton and fish. In this model, we take P(t), Z(t), and F(t) as the population densities of the toxin-producing phytoplankton (TPP), specialist predator zooplankton, and fish. It is as-

Table 1
Biological description of parameters.

Parameter	Biological description
r_1	Growth rate of TPP.
α_1	Death rate of P(t).
β_1	Maximum capture rate of P(t) by Z(t).
β_2	Maximum conversion rate of Z(t).
γ_1	Half saturation constant.
γ_2	Half saturation constant.
a_1	Maximum capture rate of Z(t) by F(t).
a_2	Maximum conversion rate of F(t).
A_1	Additional food available for zooplankton.
r_2	Death rate of Z(t).
A_2	Additional food available for fish.
θ	Rate of toxin produced by phytoplankton.
r_3	Natural dying rate of fish.

sumed that phytoplankton is predated by zooplankton, which is a favorite food for the generalist predator fish. This mathematical model is applicable where alternative food is available for both zooplankton and fish species [35–37]. This tri-trophic interaction system without delay (with the biological interpretation of parameters given in Table 1) is proposed by the following set of differential equations,

$$\left. \begin{aligned} \frac{dP}{dt} &= r_1 P - \alpha_1 P^2 - \beta_1 \frac{P}{\gamma_1 + A_1 + P} Z, \\ \frac{dZ}{dt} &= \beta_1 \beta_2 \frac{P}{\gamma_1 + A_1 + P} Z - r_2 Z \\ &\quad - a_1 \frac{P}{\gamma_2 + A_2 + Z} F - \theta \frac{P}{\gamma_1 + A_1 + P} Z, \\ \frac{dF}{dt} &= -r_3 F + a_1 a_2 \frac{Z}{\gamma_2 + A_2 + Z} F. \end{aligned} \right\} \quad (2)$$

3. Dynamical behaviour of the plankton system

3.1. Positivity and boundedness

Theorem 3.1. *The dynamical system (2) has a unique non-negative solution with the initial values $(P(0), Z(0), F(0)) \in \mathbb{R}_+^3$, where $\mathbb{R}_+^3 = \{(\xi_1, \xi_2, \xi_3) : \xi_i \geq 0, i = 1, 2, 3\}$. Further, the set $\Gamma = \{(P(t), Z(t), F(t)) \in \mathbb{R}^3; U(t) \leq \frac{r_1^2}{\alpha_1}\}$ is invariant for all the solutions in the interior of the positive octant.*

Proof. The dynamical system (2) in the form of matrix is given by, $\frac{dH}{dt} = H(\xi)$, where $\xi = (\xi_1, \xi_2, \xi_3)^T = (P, Z, F)^T \in \mathbb{R}^3$, $H(\xi) =$

$$\begin{pmatrix} H_1(\xi) \\ H_2(\xi) \\ H_3(\xi) \end{pmatrix} = \begin{pmatrix} r_1 P - \alpha_1 P^2 - \beta_1 \frac{P}{\gamma_1 + A_1 + P} Z \\ \beta_1 \beta_2 \frac{P}{\gamma_1 + A_1 + P} Z - r_2 Z - a_1 \frac{Z}{\gamma_2 + A_2 + Z} F \\ -\theta \frac{P}{\gamma_1 + A_1 + P} Z \\ -r_3 F + a_1 a_2 \frac{Z}{\gamma_2 + A_2 + Z} F \end{pmatrix}.$$

Since, $H : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is locally Lipschitz continuous in Γ along with $\xi(0) = \xi_0 \in \mathbb{R}^3$, thus by fundamental theorem of ordinary differential equations [39], there must exist unique solution of (2) and $[H_i(\xi)]_{\xi_i(t)=0, \xi \in \mathbb{R}^3} \geq 0$, implies $\xi(t) > 0 \forall t \geq 0$. From model system

(2), $\frac{dP}{dt} \Big|_{P=0} \geq 0$, $\frac{dZ}{dt} \Big|_{Z=0} \geq 0$, and $\frac{dF}{dt} \Big|_{F=0} \geq 0$. Therefore, the system (2) has a unique positive solution. \square

Next, we claim that all these solutions are uniformly bounded in the octant Γ .

Let $U(t) = P(t) + \frac{\beta_1}{\beta_1\beta_2 - \theta} Z(t) + \frac{\beta_1}{a_2(\beta_1\beta_2 - \theta)} F(t)$,
 $\frac{dU}{dt} \leq -\alpha_1(P(t) - \frac{r_1}{\alpha_1})^2 + \frac{r_1^2}{\alpha_1} - r_1P(t) - \frac{r_2\beta_1}{(\beta_1\beta_2 - \theta)} Z(t) - \frac{r_3\beta_1}{a_2(\beta_1\beta_2 - \theta)} F(t)$,
 $\frac{dU}{dt} \leq \frac{r_1^2}{\alpha_1} - \eta U(t)$, where $\eta = \min\{r_1, \frac{r_2\beta_1}{(\beta_1\beta_2 - \theta)}, \frac{r_3\beta_1}{a_2(\beta_1\beta_2 - \theta)}\}$,
 $\frac{dU}{dt} + \eta U(t) \leq \frac{r_1^2}{\alpha_1}$, implies $0 \leq U(t) \leq \frac{r_1^2}{\alpha_1} + \frac{U(P(0), Z(0), F(0))}{e^{\eta t}}$
 (using comparison theorem of ODE [40]). As $t \rightarrow \infty$, we have,
 $U(t) \leq \frac{r_1^2}{\alpha_1}$, which implies that the solutions are bounded for $0 \leq U(t) \leq \frac{r_1^2}{\alpha_1}$. Therefore, all the solutions of the given plankton system are lies in the octant,
 $\Gamma = \{(P(t), Z(t), F(t)) \in R^{+3}; U(t) \leq \frac{r_1^2}{\alpha_1} + \epsilon\}, \forall \epsilon > 0$.

4. Stability analysis

In this section, we determine the stability of the dynamical system (2) about all equilibria. Firstly, we discuss the existence of all steady states of the system and survival of the plankton fish species in the closed first octant $R^3_+ = \{(P, Z, F) : P \geq 0, Z \geq 0, F \geq 0\}$ for the set of parameters $[S_1]$: $r_1 = 2, \alpha_1 = 0.05, \beta_1 = 1, \gamma_1 = 1, A_1 = 35, r_2 = 1, \beta_2 = 3, a_1 = 1.5, \gamma_2 = 1, A_2 = 9, \theta = 1, r_3 = 0.7$, and $a_2 = 1.3333$.

4.1. Existence of equilibria

Lemma 4.1. *The positive interior equilibrium*

$V_*(P_*, Z_*, F_*)$ exists if (i) $a_1a_2 > r_3$, (ii) $G(0) = \beta_1Z_* - r_1(\gamma_1 + A_1) < 0$, and (iii) $(\beta_1\beta_2 - \theta)P_* > r_2(\gamma_1 + A_1 + P_*)$ hold true.

Proof. We obtain $Z_* = \frac{r_3(\gamma_2 + A_2)}{(a_1a_2 - r_3)}$ from given dynamical system, which exists if $a_1a_2 > r_3$. Next, substituting Z_* in first equation of given dynamics, we get $G(P_*) = \alpha_1P_*^2 + (\alpha_1\gamma_1 + \alpha_1A_1 - r_1)P_* + \beta_1Z_* - r_1(\gamma_1 + A_1)$, which has a positive root P_* if $G(0) = \beta_1Z_* - r_1(\gamma_1 + A_1) < 0$. We obtain $F_* = \frac{((\beta_1\beta_2 - \theta)P_* - r_2((\gamma_1 + A_1 + P_*)))(\gamma_2 + A_2 + Z_*)}{(\gamma_1 + A_1 + P_*)a_1}$, which exists if $(\beta_1\beta_2 - \theta)P_* > r_2((\gamma_1 + A_1 + P_*))$. □

Numerically, for a set of parameters $[S_1]$, we obtain $V_*(38.5384, 5.3852, 0.3516)$. The existence conditions of V_* are valid as (i) $a_1a_2 > r_3$ ($0.8000 > 0.0001$) (ii) $G(0) = \beta_1Z_* - r_1(\gamma_1 + A_1) < 0$ ($-6 < 0$), and (iii) $(\beta_1\beta_2 - \theta)P_* > r_2(\gamma_1 + A_1 + P_*)$ ($1.20013 > 0.44002$).

Remark 4.1. The existence of the interior equilibrium V_* indicates the co-existence of plankton-fish species at the level $V_*(38.5384, 5.3852, 0.3516)$.

Lemma 4.2. *The fish free equilibrium $V_3(P_3, Z_3, 0)$ exists if (i) $a_1a_2 > r_3$ and (ii) $g(0) = \beta_1Z_3 - r_1(\gamma_1 + A_1) < 0$ hold good.*

Proof. We obtain $Z_3 = \frac{r_3(\gamma_2 + A_2)}{(a_1a_2 - r_3)}$, which exists if $a_1a_2 > r_3$. Further, substituting the value of Z_3 in first equation of given dynamical system (2), we determine a quadratic equation $g(P_3) = \alpha_1P_3^2 + (\alpha_1\gamma_1 + \alpha_1A_1 - r_1)P_3 + \beta_1Z_3 - r_1(\gamma_1 + A_1)$, which admits a positive root P_3 , if $g(0) = \beta_1Z_3 - r_1(\gamma_1 + A_1) < 0$. □

Numerically, taking $r_3 = 1.99$ in $[S_1]$, the steady state $V_3(35.9944, 14.4007, 0)$ exists as the conditions (i) $a_1a_2 > r_3$

Table 2
Survival of plankton species in the absence and presence of fish biomass.

Absence of fish	Presence of fish
$P_3 = 35.9944$	$P_* = 38.5384$
$Z_3 = 14.4007$	$Z_* = 5.3852$

Table 3
Survival of fish species in the absence and presence of zooplankton ($Z(t)$).

Presence of $Z(t)$	Absence of $Z(t)$
$F_* = 0.3516$	$F_2 = 0.1420$

Table 4
Comparison of phytoplankton in the absence and presence of predators.

Availability of predators	Biomass of $P(t)$
In absence of $Z(t)$ and $F(t)$	$P_1 = 39.9985$
Only $Z(t)$ present	$P_3 = 35.9944$
In presence of $Z(t)$ and $F(t)$	$P_* = 38.5384$

($2 > 1.99$) and (ii) $g(0) = \beta_1Z_3 - r_1(\gamma_1 + A_1) < 0$ ($g(0) = -57.59930 < 0$) are numerically valid.

Remark 4.2. Biologically, the existence of V_3 shows that in the absence of fish species, zooplankton biomass survives at a high level and phytoplankton biomass at a low level, compared to its presence at V_* as shown in Table 2.

Lemma 4.3. *The axial steady state*

$V_2\left(\frac{r_1}{\alpha_1}, 0, \frac{((\beta_1\beta_2 - \theta - r_2)r_1 - r_2\alpha_1(\gamma_1 + A_1))(\gamma_2 + A_2)}{a_1((\gamma_1 + A_1)\alpha_1 + r_1)}\right)$ exists if $[O_1]: \beta_1\beta_2r_1 > \theta r_1 + r_2((\gamma_1 + A_1)\alpha_1 + r_1)$ holds true.

Numerically, taking $\beta_1 = 1.2, A_1 = 3, r_2 = 2$, and $r_3 = 0.1$ in $[S_1]$ (other parameters are same), the zooplankton free equilibrium $V_2(39.9985, 0, 0.1420)$ exists, as its existence condition $[O_1]: \beta_1\beta_2r_1 > \theta r_1 + r_2((\gamma_1 + A_1)\alpha_1 + r_1)$ ($7.2 > 6.400$) is satisfied.

Remark 4.3. Ecologically, the significance of the equilibrium state V_2 lies in the fact that the fish population can survive on available additional food at level $F_2 = 0.1420$, compared to its survival $F_* = 0.3516$ in the presence of its prey and never extinct (see Table 3).

Lemma 4.4. *The zero equilibrium $V_0(0, 0, 0)$ and trivial steady state $V_1\left(\frac{r_1}{\alpha_1}, 0, 0\right)$ always exist.*

Numerically, taking $\theta = 1.5$ in $[S_1]$, we obtain the predator free steady state $V_1(39.9985, 0, 0)$.

Remark 4.4. The existence of V_1 conveys that the phytoplankton biomass ($P_1 = 39.9985$) survive at a high level in the absence of both predators than their presence (see Table 4).

4.2. Stability of equilibrium points

Lemma 4.5. *The zero equilibrium V_0 is unstable as $r_1 = 0.5$ is a positive eigen value of the corresponding variational matrix.*

Lemma 4.6. *The predator free equilibrium V_1 remains stable if the condition*

$$G_1: A_1 > \frac{(\beta_1\beta_2 - \theta)r_1 - r_2(\gamma_1\alpha_1 + r_1)}{r_2\alpha_1} \text{ holds good.}$$

Numerically, taking $\theta = 1.5$ in $[S_1]$, the given system (2) is locally asymptotically stable (LAS) as (1) $G_1:(9 > 0.0048)$ is satis-

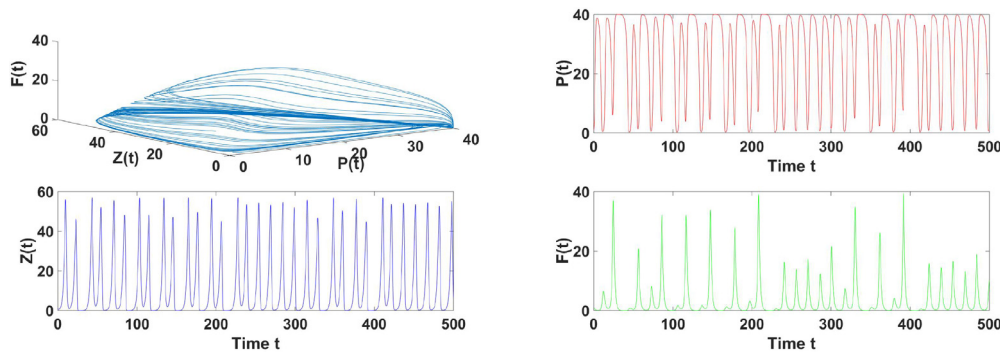


Fig. 1. Time series graph of chaotic behavior of the given system (2) using $[S_2]$.

fied. (2) The eigen values of the corresponding variational matrix around V_1 are -0.4999 , -0.0707 , and -0.0200 .

Lemma 4.7. The zooplankton free steady state V_2 remains stable as long as the condition G_2 holds true, where G_2 : (i) $r_1 < 2\alpha_1 P_2$ and (ii) $A_2 - \frac{(\beta_1 \beta_2 - \theta) P_2 + (a_1 F_2 - r_2 \gamma_2)(\gamma_1 + A_1 + P_2)}{(\gamma_1 + A_1 + P_2) r_2 - (\beta_1 \beta_2 - \theta) P_2} > 0$.

Numerically, taking $\beta_1 = 1.2$, $A_1 = 3$, $r_2 = 2$, $r_3 = 0.1$ and other parameters as in $[S_1]$, the system is LAS around V_2 as;

1. G_2 : (i) $r_1 < 2\alpha_1 P_2$ ($0.5 < 1.2001$) and (ii) $A_2 - \frac{(\beta_1 \beta_2 - \theta) P_2 + (a_1 F_2 - r_2 \gamma_2)(\gamma_1 + A_1 + P_2)}{(\gamma_1 + A_1 + P_2) r_2 - (\beta_1 \beta_2 - \theta) P_2} > 0$ ($64.17790 > 0$).
2. All the eigen values of the jacobian matrix corresponding to V_2 are negative i.e. -0.60013 , -0.00010 , and -0.07872 .

Lemma 4.8. The fish free equilibrium V_3 remains locally asymptotically stable as long as the condition G_3 : $E_1 E_2 - E_3 > 0$ and $E_i > 0$, $i = 1, 3$ (Routh-Hurwitz criterion) is satisfied.

The values of E_i 's are given in Appendix A.

Numerically, the steady state V_3 is LAS as G_3 : $E_1 E_2 - E_3 = 0.00022 > 0$, $E_1 = 0.0906 > 0$, and $E_3 = 0.00063 > 0$ holds true (using $r_3 = 1.99$ in $[S_1]$).

Lemma 4.9. The interior equilibrium V_* remains LAS if $F_1 F_2 - F_3 > 0$ and $F_i > 0 \forall i = 1, 3$ (Routh-Hurwitz criterion).

The values of F_i 's are given in Appendix A.

Numerically, for the set of parameters $[S_1]$ the interior equilibrium V_* is locally asymptotically stable as (1) $F_1 F_2 - F_3 = 0.02562 > 0$, $F_1 = 1.876910 > 0$, and $F_3 = 0.0294 > 0$. (2) The eigen values of the corresponding variational matrix around V_* are $-0.00363 \pm 0.12547i$ and -1.86964 .

5. Analysis of chaotic dynamics

In this section, we will introduce certain criteria to control the complexity of the given plankton system w.r.t. some internal and external parameters. For that we consider a set of parameters $[S_2]$: taking $A_1 = 7$ and rest of the parametric values are same as in $[S_1]$.

5.1. Dissipativity condition [41,42]

Let U be the volume of the moving space of the phase space motion. To verify the dissipativity condition, we consider the following function, $\nabla(U) = \frac{dP}{dt} + \frac{dZ}{dt} + \frac{dF}{dt}$.

After some simple mathematical calculations at V_* , we get

$$\nabla(U) \leq -(2\alpha_1 P_* + \frac{\beta_1(\alpha_1 + A_1)}{(\gamma_1 + A_1 + P_*)^2} + r_2 + \frac{a_1(\gamma_2 + A_2)}{(\gamma_2 + A_2 + Z_*)^2} + r_3) < 0.$$

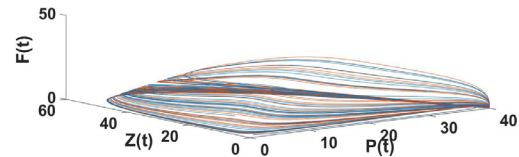


Fig. 2. Phase portrait showing sensitive dependence to initial condition of the trajectories of chaotic system using $[S_2]$.

As $2\alpha_1 P_* + \frac{\beta_1(\alpha_1 + A_1)}{(\gamma_1 + A_1 + P_*)^2} + r_2 + \frac{a_1(\gamma_2 + A_2)}{(\gamma_2 + A_2 + Z_*)^2} + r_3 > 0$ and $t \rightarrow \infty$, each volume containing the trajectories of dynamics (2) shrinks to zero at an exponential rate. Thus, all the dynamical orbits are confined to a subset of zero volume and the asymptotic behavior settles onto a chaotic attractor in the three-dimensional phase space Fig. 1 using $[S_2]$.

5.2. Sensitivity to initial conditions

In Fig. 3, the time-series graph shows the chaotic behavior of state variables $P(t)$, $Z(t)$, and $F(t)$. Moreover, the system (2) is chaotic as it is sensitive to the initial condition, which is one of the most visible sign of chaotic behavior of any dynamical system. We have observed from the phase portrait (Fig. 2) that both curves follow the same path for some time, but after that, these trajectories divert from each other. Indeed, the two-time series graph of each state variable in the system (2) about neighboring initial conditions is shown in Fig. 3. The first time-series graph of Fig. 3 shows that both curves perfectly superimposed up to 50 iterations and after that, the curve (red line) drawn with the initial condition $[1.1,1,1]$ divert from the path drawn by the curve (blue line) with the initial condition $[1,1,1]$. Similarly, second and third graph of Fig. 3 show the sensitivity of the system (2) about the initial condition for the set of parameters $[S_2]$.

5.3. Analysis of Lyapunov exponents

The most effective way to detect the chaos in any dynamical system is to check the growth or decline rate of the small perturbations along the axis of the phase diagram through the Lyapunov exponent. For the given dynamical system (2), we have estimated three Lyapunov exponents using MATLAB software and the integrator ode87. Fig. 4 exhibits the obtained Lyapunov spectrum of the chaotic system. The exponent $\lambda_1 = 0.0293$ shows the increase in the expansion degree of chaotic attractor, where the smallest exponent $\lambda_3 = -1.0223$ exhibits the increase in contraction degree of attractor in the phase diagram. The critical nature of this chaotic attractor is translated by the exponent $\lambda_2 = -0.0016$. The Kaplan-Yorke dimension of the chaotic system dynamics (2), which shows the complexity of the strange attractor, is given by;

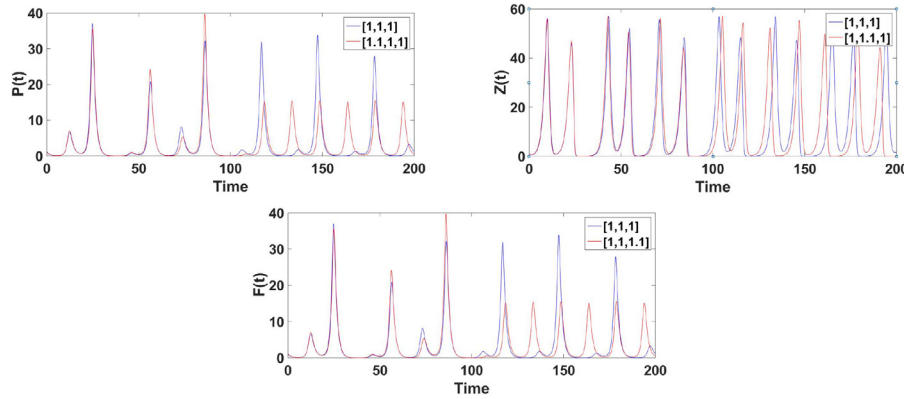


Fig. 3. Sensitive dependence to initial condition of the state trajectories $P(t)$, $Z(t)$, and $F(t)$ of chaotic system using $[S_2]$.

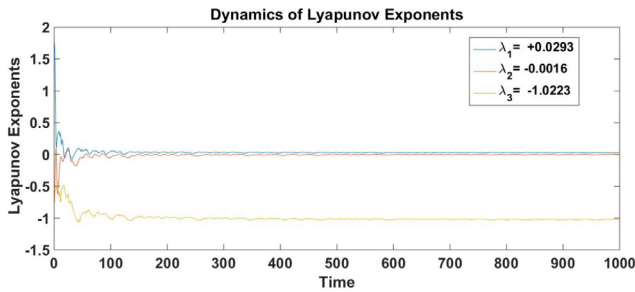


Fig. 4. Lyapunov spectrum of system (2) using $[S_2]$.

$$D_L = 2 + \frac{\lambda_1 + \lambda_2}{|\lambda_3|} = 2.0161.$$

Therefore, the dynamical system (2) generates chaotic behaviors characterized by fractional-order dimension.

5.4. Analysis of system dynamics with respect to available additional food to both predators

Firstly, we prove that the system (2) observes Hopf-bifurcation with the occurrence of periodic oscillations w.r.t. available additional food to zooplankton (A_1).

Lemma 5.1. *The chaotic dynamical system enters into Hopf-bifurcation around the interior point V_* as A_1 passes through its critical value A_1^* under the following conditions,*

1. $F_1(A_1^*) > 0, i=1,3, F_1(A_1^*)F_2(A_1^*) - F_3(A_1^*) = 0,$
2. $(F_1(A_1^*)F_2(A_1^*))' \neq (F_3(A_1^*))'.$

Proof. We consider A_1 as a bifurcation parameter, the given plankton system shows excitability if there exists a critical value A_1^* of A_1 such that $F_1(A_1^*)F_2(A_1^*) - F_3(A_1^*) = 0$. Thus, the characteristic equation

$$\lambda^3 + F_1\lambda^2 + F_2\lambda + F_3 = 0, \tag{3}$$

must have of the following form at $A_1 = A_1^*$

$$(\lambda^2(A_1^*) + F_2(A_1^*))(\lambda(A_1^*) + F_1(A_1^*)) = 0, \tag{4}$$

The Eqn. 4 clearly has roots $-F_1(A_1^*)$ and $\pm i\sqrt{F_2(A_1^*)}$. But, in general, $\lambda_1(A_1) = u(A_1) + iv(A_1), \lambda_2(A_1) = u(A_1) - iv(A_1),$ and $\lambda_3(A_1) = F_1(A_1)$. Substituting values of $\lambda_i, i = 1, 2$ in (3) and calculating the derivatives, we get

$$\begin{cases} M_1(A_1)u'(A_1) - M_2(A_1)v'(A_1) + M_3(A_1) = 0, \\ M_1(A_1)u'(A_1) + M_2(A_1)v'(A_1) + M_4(A_1) = 0. \end{cases} \tag{5}$$

Where $M_1(A_1) = 3u^2(A_1) + 2F_1(A_1)u(A_1) + F_2(A_1) - 3v^2(A_1),$

$$\begin{aligned} M_2(A_1) &= 6u(A_1)v(A_1) + 2F_1(A_1)v(A_1), \\ M_3(A_1) &= u^2(A_1)F_1'(A_1) + F_2'(A_1)u(A_1) + F_3'(A_1) - F_1'(A_1)v^2(A_1), \end{aligned}$$

and

$$M_4(A_1) = 2u(A_1)v(A_1)F_1'(A_1) + F_2'(A_1)v(A_1).$$

Taking $u(A_1^*) = 0$ and $v(A_1^*) = \sqrt{F_2(A_1^*)}$, we obtain

$$M_1(A_1^*) = -2F_2(A_1^*), \quad M_2(A_1^*) = 2F_1(A_1^*)\sqrt{F_2(A_1^*)}, \quad M_3(A_1^*) = F_3'(A_1^*) - F_1'(A_1^*)F_2(A_1^*),$$

$$M_4(A_1^*) = F_2'(A_1^*)\sqrt{F_2(A_1^*)}.$$

Solving (5) for $u'(A_1)$, we get

$$(u'(A_1))_{A_1=A_1^*} = -\frac{M_2(A_1^*)M_4(A_1^*) + M_1(A_1^*)M_3(A_1^*)}{M_1^2(A_1^*) + M_2^2(A_1^*)}$$

$$= \frac{-(F_1(A_1^*)F_2(A_1^*))' - F_3'(A_1^*)}{2(F_1(A_1^*))^2 + F_2(A_1^*)} \neq 0 \text{ (using given hypothesis). } \square$$

Numerically, $\left[\frac{d(u)}{dA_1} \right]_{A_1=33.5} = -0.176479 \neq 0.$

Therefore, the transversality condition holds, which results in the occurrence of Hopf-bifurcation at $A_1 = A_1^* = 33.5$.

Now, we discuss the chaos control strategies of the system (2) using A_1 and A_2 .

The chaotic behavior in natural population is very commonly studied from three-four decades [9,14,19,26,43–45]. These studies targeted mainly aquatic species because the marine world plays a significant role in sustaining life in the biosphere. Many scientists suggested various techniques (feedback and non-feedback) and distinct parameters to suppress chaotic dynamics and get stable trajectories [9,14–17]. In this subsection, we utilize the available additional food to zooplankton and fish as chaos control parameters.

- We have observed that as the natural supply of additional food for zooplankton (A_1) increases in $[S_2]$, the chaotic system turns to a stable one. The given system shows chaotic trajectories for the set of parameters $[S_2]$ (Fig. 6 a), it changes its behavior from chaotic to quasi-periodic at $A_1 = 14$ (Fig. 6 b), double periodic at $A_1 = 14.8$ (Fig. 6 c), and turns to single periodic at $A_1 = 28$ (Fig. 6 d). The system observes periodic oscillations with the occurrence of Hopf-bifurcation at $A_1 = 33.5$ (Fig. 6 e) and switches to stability at $A_1 = 34$ (Fig. 6 f). In Fig. 5, the bifurcation diagrams of $P(t)$, $Z(t)$, and $F(t)$ reveal rich dynamical behavior in the system including chaos.
- Fig. 7 shows that chaos disappears with the occurrence of a stable limit cycle as the available alternative food for fish (A_2) varies in $[S_2]$. We have observed by simulation that the chaotic dynamics (2) switch to multiple periods at $A_2 = 12$ (Fig. 7 b), countable limit cycles at $A_2 = 20$ (Fig. 7 c), and settle down at a stable limit cycle at $A_2 = 25$ (Fig. 7 d).

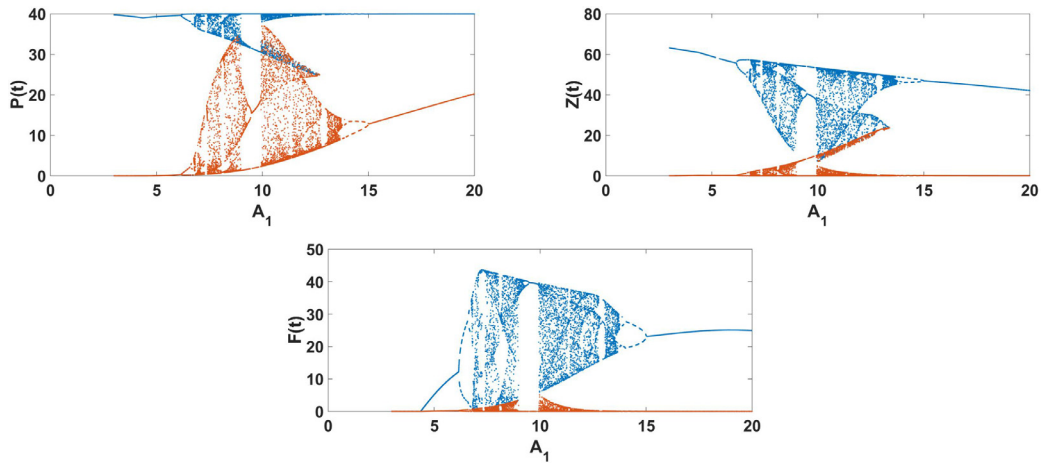


Fig. 5. Bifurcation diagrams with respect to A_1 of the system (2) using $[S_2]$.

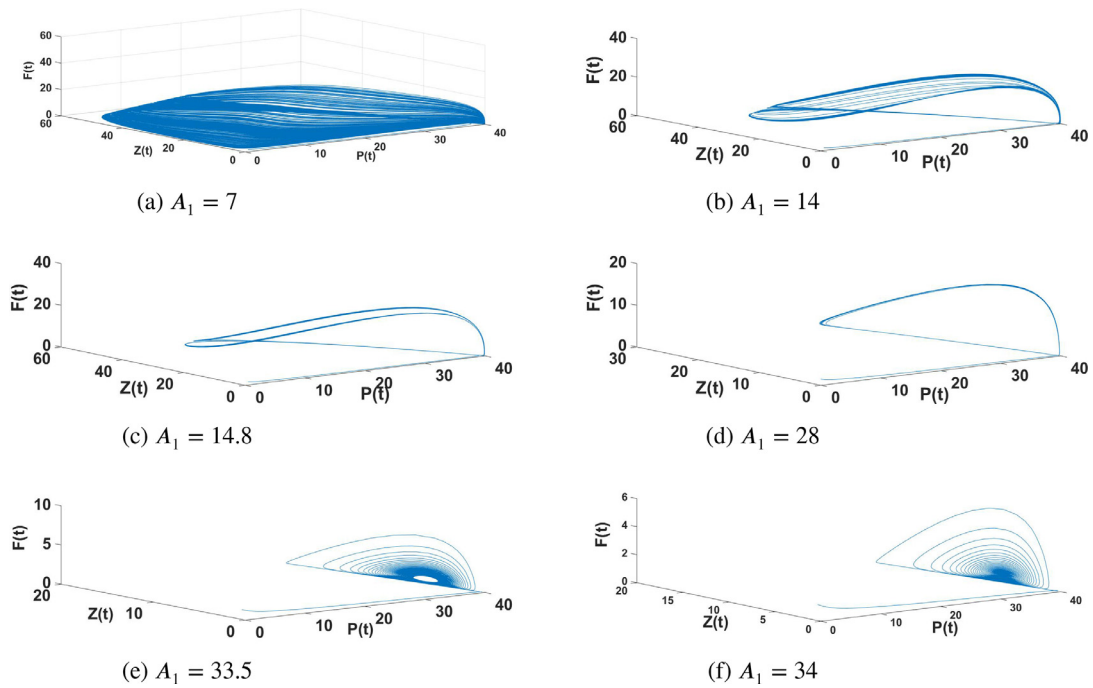


Fig. 6. Behavior of the system with respect to available additional food for zooplankton A_1 using $[S_2]$.

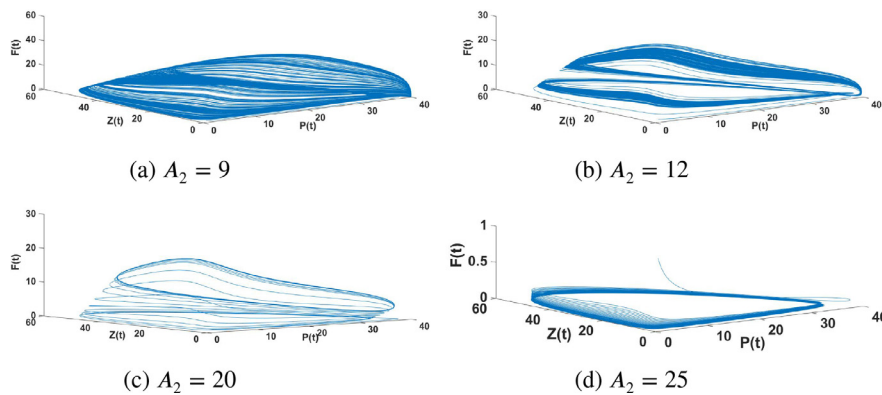


Fig. 7. Behavior of the system with respect to available additional food for fish (A_2) using $[S_2]$.

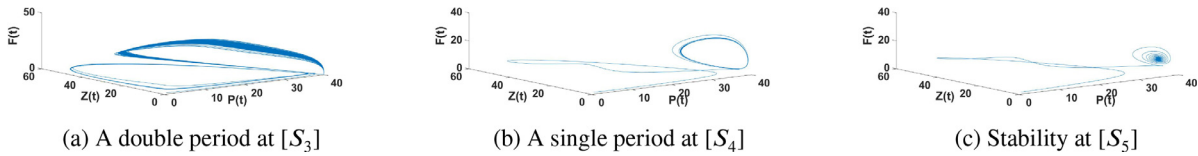


Fig. 8. Existence of stable, single periodic and double periodic orbits.

5.5. Chaos control using sinusoidal force

The impact of seasonality, noise, periodic forcing on population dynamics have been studied by many ecologists [46–50]. They have observed that the system becomes chaotic due to the seasonality or external powers (sinusoidal force). In the previous two three decades, several studies are available which investigates the chaos control strategies [18,19,51]. But the impact of seasonal forces for controlling chaos in the marine population is rarely seen by us. In this subsection, we have perturbed the system (2) by imposing the sinusoidal force on population dynamics of zooplankton species. Thus, a weak seasonal periodic force $h(t)$ is applied for controlling the unpredictable behavior of the given dynamical system, where $h(t) = J_1 + J_2 \sin(\alpha t)$, in which J_1 is a constant bias, J_2 , and α denote the amplitude and frequency of the external periodic force.

$$\left. \begin{aligned} \frac{dP}{dt} &= r_1P - \alpha_1P^2 - \beta_1 \frac{P}{\gamma_1 + A_1 + P} Z, \\ \frac{dZ}{dt} &= \beta_1\beta_2 \frac{P}{\gamma_1 + A_1 + P} Z - r_2Z - a_1 \frac{Z}{\gamma_2 + A_2 + Z} F(t) \\ &\quad - \theta \frac{P}{\gamma_1 + A_1 + P} Z + h(t), \\ \frac{dF}{dt} &= -r_3F + a_1a_2 \frac{Z}{\gamma_2 + A_2 + Z} F. \end{aligned} \right\} \quad (6)$$

We have carried out numerical simulation of the controlled system (6) using following set of parameters; $[S_3]$: $[S_2]$, $J_1 = 0.05$, $J_2 = 0.05$, and $\alpha = 22/7$.

$[S_4]$: $[S_2]$, $J_1 = 1$, $J_2 = 1$, and $\alpha = 22/7$.

$[S_5]$: $[S_2]$, $J_1 = 1$, $J_2 = 2$, and $\alpha = 22/7$.

We have observed that the controlled system (6) converts the chaotic trajectories of given dynamics (2) to double periodic orbit at $[S_3]$ (Fig. 8 a), single periodic at $[S_4]$ (Fig. 8b), and settled down to stable focus around (37.6431, 5.3866, 10.4678) for the set of parameters $[S_5]$ (Fig. 8c).

6. Delayed model system

In this section, we include a predation delay in model system (2) and obtained the following delayed dynamics.

$$\left. \begin{aligned} \frac{dP}{dt} &= r_1P - \alpha_1P^2 - \beta_1 \frac{P}{\gamma_1 + A_1 + P} Z, \\ \frac{dZ}{dt} &= \beta_1\beta_2 \frac{P}{\gamma_1 + A_1 + P} Z - r_2Z \\ &\quad - a_1 \frac{Z}{\gamma_2 + A_2 + Z} F(t - \tau) - \theta \frac{P}{\gamma_1 + A_1 + P} Z, \\ \frac{dF}{dt} &= -r_3F + a_1a_2 \frac{Z}{\gamma_2 + A_2 + Z} F. \end{aligned} \right\} \quad (7)$$

6.1. Positivity and boundedness

Theorem 6.1. The positive interior equilibrium $V_*(P_*, Z_*, F_*)$ of the dynamical system (7) is invariant in positive quadrant.

Proof. We want to show that for all $0 \leq t < T^*$, ($T^* > 0$), $P(t) > 0$, $Z(t) > 0$, and $F(t) > 0$ with the initial conditions $P(0) > 0$,

$Z(0) > 0$, and $F(0) > 0$, otherwise, it can be assumed that \exists a K , where $0 < K < T^*$ such that $\forall t \in [0, K)$, $P(t) > 0$, $Z(t) > 0$, and $F(t) > 0$, and one of $P(K)$, $Z(K)$, and $F(K)$ is zero for any $t \in [-\tau, K)$. Integrating the given model system (7), we have $P(K) = P(0)e^{\int_0^t (r_1 - \alpha_1 P - \beta_1 \frac{Z}{\gamma_1 + A_1 + P}) ds}$, $Z(K) = Z(0)e^{\int_0^t (W) ds}$, where $W = \beta_1\beta_2 \frac{P}{\gamma_1 + A_1 + P} - r_2 - a_1 \frac{Z}{\gamma_2 + A_2 + Z} F(t - \tau) - \theta \frac{P}{\gamma_1 + A_1 + P}$, and $F(K) = F(0)e^{\int_0^t (-r_3 + a_1a_2 \frac{Z}{\gamma_2 + A_2 + Z}) ds}$. Since $P(t)$, $Z(t)$, and $F(t)$ are all continuous functions in $[-\tau, K)$, there exist

$S > 0$ such that $P(K) = P(0)e^{\int_0^t (r_1 - \alpha_1 P - \beta_1 \frac{Z}{\gamma_1 + A_1 + P}) ds} > P(0)e^{-KS}$, $Z(K) = Z(0)e^{\int_0^t (W) ds} > Z(0)e^{-KS}$, and $F(K) = F(0)e^{\int_0^t (-r_3 + a_1a_2 \frac{Z}{\gamma_2 + A_2 + Z}) ds} > F(0)e^{-KS}$, $\forall t \in [-\tau, K)$. Taking $t \rightarrow K$, we get $P(K) > 0$, $Z(K) > 0$, and $F(K) > 0$, a contradiction. Thus $P(t) > 0$, $Z(t) > 0$, and $F(t) > 0$ for any $0 \leq t < T^*$. \square

6.2. Hopf-bifurcation analysis of the system with delay

See Appendix B.

7. Chaos control through predation delay

Many mathematical studies [18,20–22,26,29,52] from the very first research [53] on time delay prove that the inclusion of time lag in ecological models results in occurrence of Hopf-bifurcation with periodic oscillations. But, Thakur et al. [34] have analyzed in their recent study that time delay plays a vital role in controlling chaotic plankton system. They have investigated that as the discrete delay τ increases, the chaotic plankton fish dynamics exhibits periodic oscillations of order two, then reduces to limit cycle, and finally shows stable focus. We have also showed numerically that the predation delay can stabilize the chaotic plankton-fish dynamics. Fig. 10 (using $[S_2]$) shows that in the absence of predation delay τ and at $\tau = 0.0001$, the model system (7) shows chaotic behavior. As the predation delay increases from 0.0001 to 0.03, the chaotic dynamics convert into multiple trajectories, becomes double periodic for $\tau = 0.05$, and shows a stable limit cycle at $\tau = 1.1$.

7.1. Chaos control through delayed feedback scheme

Delayed feedback control is a well-documented mechanism that has been implemented in various spatially extended systems of optics, chemical sciences, biology, and neurology [48,49]. But the impact of time delay on the marine ecosystem through delayed feedback technique is less known. In this subsection, we have constructed a controlled plankton-fish dynamics (8) to eliminate chaos and stabilize the unstable periodic orbits of period τ of the system

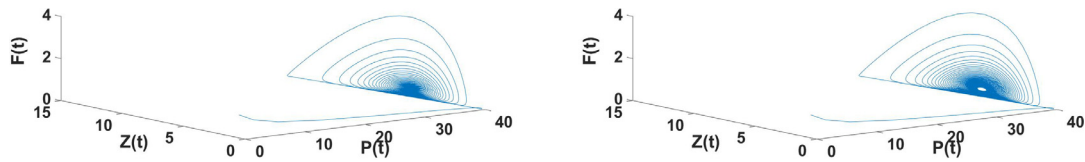


Fig. 9. Behavior of the system at different values of τ .

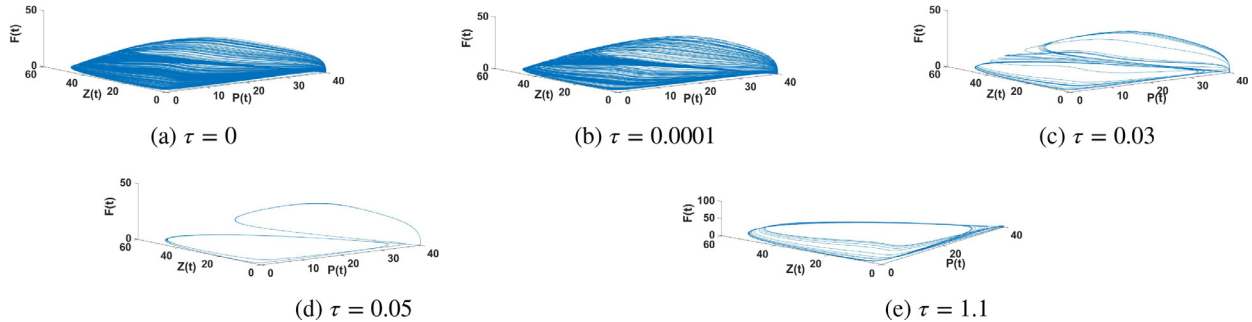


Fig. 10. Control of chaos through predation delay τ .

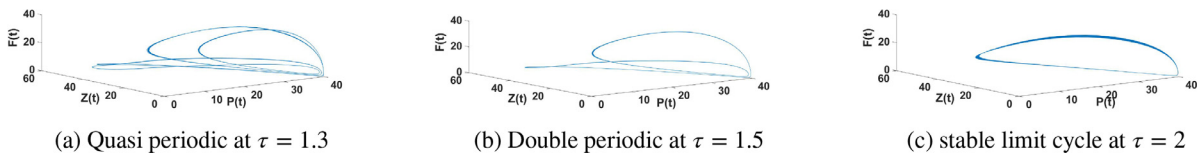


Fig. 11. Dynamics of the system (8) using $[S_6]$.

(2). The controlled dynamical system is as given below,

$$\begin{cases} \frac{dP}{dt} = r_1P - \alpha_1P^2 - \beta_1 \frac{P}{\gamma_1 + A_1 + P}Z - K_1(P(t) - P(t - \tau)), \\ \frac{dZ}{dt} = \beta_1\beta_2 \frac{P}{\gamma_1 + A_1 + P}Z - r_2Z - a_1 \frac{Z}{\gamma_2 + A_2 + Z}F(t) - \theta \frac{P}{\gamma_1 + A_1 + P}Z - K_2(Z(t) - Z(t - \tau)), \\ \frac{dF}{dt} = -r_3F + a_1a_2 \frac{Z}{\gamma_2 + A_2 + Z}F - K_3(F(t) - F(t - \tau)). \end{cases} \quad (8)$$

Here, $diag(K_1, K_2, K_3)$ is the feedback matrix, and τ is taken to be the same as the period of the target unstable periodic orbits. Firstly, we consider set of parameters $[S_6]$: $[S_2]$, $K_1 = 1$, $K_2 = 1$, $K_3 = 1$, and $\tau = 1.3$, for which the system (8) shows periodic oscillations of period four, as τ increases from 1.3 to 1.5, it converts into double period, and finally becomes stable around stable limit cycle at $\tau = 2$ and for higher values of τ (Fig. 11).

8. Conclusion

In this article, we proposed some ecological models to study the chaotic behavior of plankton dynamics. We observed that chaos can be controlled by varying factors/parameters, viz., available additional food, seasonality, predation delay etc. Analytically, we studied stability analysis of delayed and non-delayed plankton dynamics. The Hopf-bifurcation analysis is performed using A_1 and predation delay as bifurcation parameters. Ecologically, we have observed the following implications of our study.

- The available additional food for zooplankton A_1 has a significant role in the dynamics of the proposed dynamical system. It is observed from Fig. 6 that its suitable availability stabilizes the chaotic behavior of the dynamical system (2). The given system shows chaotic behavior at $[S_2]$, quasi periodic at $A_1 = 14$, dou-

ble periodic at $A_1 = 14.8$, Hopf-bifurcation at $A_1 = 33.5$, and becomes stable at $A_1 = 35$. The bifurcation diagrams in Fig. 5 confirm the rich dynamical behavior of the system (2) at different levels of A_1 .

- The additional food for fish A_2 also has the similar effects on the dynamics of the proposed dynamical system. From Fig. 7, it is observed that as (A_2) increases from 9 to 12 in $[S_2]$ the system changes its behavior from chaos to multiple periods, countable limit cycles at $A_2 = 20$, and a stable limit cycle at $A_2 = 25$.
- The results of our study reveal that the weak sinusoidal force can convert the chaotic behavior of the system to double periodic orbit using $[S_3]$, single periodic orbit using $[S_4]$, and finally system becomes stable for the set of parameters $[S_5]$ as presented in Fig. 8.
- We have observed from Fig. 9 that predation delay not only unstabilizes the stable dynamics (2) but also stabilizes the chaotic system as shown in Fig. 10.
- A chaos control delayed feedback scheme (8) has been presented to suppress the chaotic nature of plankton dynamics (2). Fig. 11 shows that as τ increases in $[S_6]$, the system (8) shows periodic trajectories of period four, then of the period two, and finally the system becomes stable around the stable limit cycle.

Thus, it can be concluded that the present biological models exhibit various intervals of internal parameters and external factors that shows rich dynamics and control of chaos in the plankton-fish ecosystem.

Credit Author Statement

Rajinder Pal Kaur: Model formation, calculations, simulations, original manuscript preparation. Dr. Amit Sharma: Model formulation, methodology, conceptualizations, rewriting and editing. Dr. Anuj Kumar Sharma: Supervision. Dr. Govind Prasad Sahu: Simulation, manuscript revision, editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

1. The variational matrix W_* of the positive interior equilibrium V_* is given by;

$$W_* = \begin{bmatrix} A_{100} & A_{010} & A_{001} \\ B_{100} & B_{010} & B_{001} \\ C_{100} & C_{010} & C_{001} \end{bmatrix}, \quad \text{where} \quad A_{100} = r_1 - 2\alpha_1 P_* - \frac{\beta_1 Z_* (\gamma_1 + A_1)}{(\gamma_1 + A_1 + P_*)^2}, \quad A_{010} = -\frac{\beta_1 P_*}{(\gamma_1 + A_1 + P_*)}, \quad A_{001} = 0, \quad B_{100} = \frac{(\beta_1 \beta_2 - \theta) (\gamma_1 + A_1) Z_*}{(\gamma_1 + A_1 + P_*)^2}, \quad B_{010} = -r_2 + \frac{(\beta_1 \beta_2 - \theta) P_*}{(\gamma_1 + A_1 + P_*)} - \frac{(a_1 (\gamma_2 + A_2) F_*)}{(\gamma_2 + A_2 + Z_*)^2}, \quad B_{001} = -\frac{a_1 Z_*}{(\gamma_2 + A_2 + Z_*)}, \quad C_{100} = 0, \quad C_{010} = \frac{a_1 a_2 (\gamma_2 + A_2) F_*}{(\gamma_2 + A_2 + Z_*)^2}, \quad C_{001} = -r_3 + \frac{a_1 a_2 Z_*}{(\gamma_2 + A_2 + Z_*)}.$$

The characteristic equation of the variational matrix W_* w.r.t. V_* is $\lambda^3 + F_1 \lambda^2 + F_2 \lambda + F_3 = 0$, where $F_1 = -(A_{100} + B_{010} + C_{001})$, $F_2 = -A_{010} B_{100} - A_{001} C_{100} - C_{010} B_{001} + B_{010} C_{001} + A_{100} B_{010} + A_{100} C_{001}$, $F_3 = A_{100} C_{010} B_{001} + A_{010} B_{100} C_{001} + A_{001} C_{100} B_{010} - A_{100} B_{010} C_{001} - A_{010} C_{100} B_{001} - A_{001} B_{100} C_{010}$.

Now, using Routh-Hurwitz criterion, the variation matrix W_* has negative eigen values or eigen values with negative real parts if $F_1 F_2 - F_3 > 0$ and $F_i > 0$ for $i = 1, 3$ holds true around V_* .

2. The variational matrix U_* of the positive interior equilibrium V_3 is given by;

$$U_* = \begin{bmatrix} d_{100} & d_{010} & d_{001} \\ e_{100} & e_{010} & e_{001} \\ f_{100} & f_{010} & f_{001} \end{bmatrix}, \quad \text{where} \quad d_{100} = r_1 - 2\alpha_1 P_3 - \frac{(\beta_1 Z_3 (\gamma_1 + A_1))}{(\gamma_1 + A_1 + P_3)^2}, \quad d_{010} = -\frac{(\beta_1 P_3)}{(\gamma_1 + A_1 + P_3)}, \quad d_{001} = 0, \quad e_{100} = \frac{(\beta_1 \beta_2 - \theta) * (\gamma_1 + A_1) * Z_3}{(\gamma_1 + A_1 + P_3)^2}, \quad e_{010} = -r_2 + \frac{(\beta_1 \beta_2 - \theta) P_3}{(\gamma_1 + A_1 + P_3)}, \quad e_{001} = -\frac{a_1 * Z_3}{(\gamma_2 + A_2 + Z_3)}, \quad f_{100} = 0, \quad f_{010} = 0, \quad \text{and} \quad f_{001} = -r_3 + \frac{a_1 a_2 Z_3}{(\gamma_2 + A_2 + Z_3)}.$$

The characteristic equation of the variational matrix U_* w.r.t. V_3 is $\lambda^3 + E_1 \lambda^2 + E_2 \lambda + E_3 = 0$, where $E_1 = -(d_{100} + e_{010} + f_{001})$, $E_2 = -d_{010} e_{100} - d_{001} f_{100} - f_{010} e_{001} + e_{010} f_{001} + d_{100} z_{010} + d_{100} f_{001}$, and $E_3 = d_{100} f_{010} e_{001} + d_{010} e_{100} f_{001} + d_{001} f_{100} e_{010} - d_{100} e_{010} f_{001} - d_{010} f_{100} e_{001} - d_{001} e_{100} f_{010}$. Now, the steady state V_3 is locally asymptotically stable if $G_3: E_1 E_2 - E_3 > 0$ and $E_i > 0$ for $i = 1, 3$ hold true.

Appendix B

To determine the occurrence of Hopf-bifurcation w.r.t. time lag τ in the given dynamical system (7), we linearize the differential equations about V_* by choosing $p = P - P_*$, $z = Z - Z_*$ and $f = F - F_*$. Applying simple calculations and using theorems on delay differential equations [54], we get the characteristic equation for the system (7) at V_* as

$$\lambda^3 + D_1 \lambda^2 + D_2 \lambda + D_3 + (D_4 + D_5 \lambda) e^{-\lambda \tau} = 0, \tag{B.1}$$

where $D_1 = -a_{100} - b_{010} - c_{001}$, $D_2 = a_{100} b_{010} - a_{010} b_{100} + a_{100} c_{001} + c_{001} b_{010}$, $D_3 = -a_{100} b_{010} c_{001} + a_{010} b_{100} c_{001}$, $D_4 = a_{100} c_{010} b'_{001}$, $D_5 = b'_{001} c_{010}$

$$a_{100} = r_1 - 2\alpha_1 P_* - \frac{\beta_1 (\gamma_1 + A_1) Z_*}{(\gamma_1 + A_1 + P_*)^2}, \quad a_{010} = -\frac{\beta_1 P_*}{(\gamma_1 + A_1 + P_*)},$$

$$a_{001} = 0, \quad b_{100} = \frac{(\beta_1 \beta_2 - \theta) (\gamma_1 + A_1) Z_*}{(\gamma_1 + A_1 + P_*)^2}, \quad b_{010} = -r_2 + \frac{(\beta_1 \beta_2 - \theta) P_*}{(\gamma_1 + A_1 + P_*)} - \frac{a_1 (\gamma_2 + A_2) F_*}{(\gamma_2 + A_2 + Z_*)^2}, \quad b_{001} = 0, \quad b'_{001} = -\frac{a_1 Z_*}{(\gamma_2 + A_2 + Z_*)},$$

and $c_{010} = \frac{a_1 a_2 (\gamma_2 + A_2) F_*}{(\gamma_2 + A_2 + Z_*)^2}$, $c_{001} = -r_3 + \frac{a_1 a_2 Z_*}{(\gamma_2 + A_2 + Z_*)}$.

After some mathematical simplifications and using criteria of [30,31], we can obtain $\tau_k = \frac{1}{\omega_{10}} \arctan \frac{D_5 \omega_{10} (D_1 \omega_{10}^2 - D_3) + D_4 (D_2 \omega_{10} - \omega_{10}^3)}{(D_1 \omega_{10}^2 - D_3) - D_5 \omega_{10} (D_2 \omega_{10} - \omega_{10}^3)} + \frac{2k\pi}{\omega_{10}}$, $k = 0, 1, 2, \dots$, provided $D_4 ((D_2 - 3\omega_{10}^2) \sin \omega_{10} \tau + 2\omega_{10} D_1 \cos \omega_{10} \tau) - \omega_{10} D_5 (\omega_{10} D_5 (D_2 - 3\omega_{10}^2) \cos \omega_{10} \tau - 2\omega_{10} D_1 \sin \omega_{10} \tau + D_5) \neq 0$.

$\left(\frac{d\Re(\lambda)}{d\tau}\right)^{-1} \Big|_{\sigma=0, \tau=\tau_0} = \frac{L}{\omega_{10} (D_4^2 + \omega_{10}^2 D_5^2)} \neq 0$, where $L = D_4 ((D_2 - 3\omega_{10}^2) \sin \omega_{10} \tau + 2\omega_{10} D_1 \cos \omega_{10} \tau) - \omega_{10} D_5 (\omega_{10} D_5 (D_2 - 3\omega_{10}^2) \cos \omega_{10} \tau - 2\omega_{10} D_1 \sin \omega_{10} \tau + D_5)$, ω_{10} is purely imaginary root of (B.1), and τ_0 is the critical value of τ after which Hopf-bifurcation occurs.

Numerically, for the set of parameters [S1], the transversality condition $\left(\frac{d\Re(\lambda)}{d\tau}\right)^{-1} \Big|_{\sigma=0, \tau=\tau_0} = 0.0817 \neq 0$ holds at $\tau = 0.4$

that confirm the occurrence of Hopf bifurcation with the existence of periodic solutions (Fig. 8). The critical value of time delay for which stability exchanges takes place is $\tau_0 = 0.4$ such that V_* remains stable in $(0, 0.4)$ and bifurcation occurs for $\tau_0 \geq 0.4$.

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